A Classification of Certain Finite Double Coset Collections in the Classical Groups

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May 23, 2003

Abstract

Let G be a classical algebraic group, X a maximal rank reductive subgroup and P a parabolic subgroup. This paper classifies when $X\backslash G/P$ is finite. Finiteness is proven using geometric arguments about the action of X on subspaces of the natural module for G. Infiniteness is proven using a dimension criterion which involves root systems.

1 Statement of results

Let G be a classical algebraic group defined over an algebraically closed field, let X be a maximal rank reductive subgroup, and let P be a parabolic subgroup. The property of finiteness for $X\backslash G/P$ is preserved under taking isogenies, quotients by the center of G, connected components and conjugates (see Lemma 2.2 for a precise statement). Thus, if desired, we can specify only the Lie type of G. Similarly, we can specify only the conjugacy class of X and P; thus we usually give the Lie type of X and describe P by crossing off nodes from the Dynkin diagram for G. For the purpose of classifying finiteness, it suffices to consider only those X which are defined over $\mathbb{Z}$.

A subgroup X is spherical if $X\backslash G/B$ is finite for some (or, equivalently, for each) Borel subgroup B. For each classical group we list in Table 1 those maximal rank reductive spherical subgroups which are defined over $\mathbb{Z}$. We first describe the notation which is used for the list, and for the rest of the paper, and then describe how the list is obtained. We write $X = A_nA_mT_1$ if X is a group of Lie type $A_n + A_m$ which has a 1-dimensional central torus,

AMS subject codes: 14L, 20G15. *Keywords:* Classical groups, double cosets, finite orbits

Table 1: Maximal rank reductive spherical subgroups defined over $\mathbb{Z}$

X	$\leq$	G	X	$\leq$	G
$A_nA_mT_1$	$\leq$	A_{n+m+1}	B_nD_m	$\leq$	B_{n+m}
C_nC_m	$\leq$	C_{n+m}	$A_{n-1}T_1$	$\leq$	B_n
$C_{n-1}T_1$	$\leq$	C_n	D_nD_m	$\leq$	D_{n+m}
$A_{n-1}T_1$	$\leq$	C_n	$A_{n-1}T_1$	$\leq$	D_n

and we use similar notation for other subgroups. If G equals D_n we adopt a notational convention to distinguish between certain subgroups of the same Lie type which are not conjugate. In $G = \mathrm{SO}(V)$ any factor denoted by D_{n_1} (or SO_{2n_1}) acts as $\mathrm{SO}(V_1)$ for some decomposition $V = V_1 \perp V_2$ and any two factors denoted by $A_{n_1}T_1$ (or GL_{n_1+1}) act as GL_{n_1+1} on a pair of totally singular subspaces E and F such that $V = (E \oplus F) \perp V_2$ (in particular $\dim E = \dim F = n_1 + 1$ and E and F are in duality). We allow the notation A_0, B_0 and C_0 to denote trivial groups and D_1 to denote a group which is a 1-dimensional torus. We now describe how Table 1 is obtained. Krämer [8] classified the reductive spherical subgroups in characteristic 0. The subgroups on Krämer's list were shown to be spherical in all characteristics by Brundan [3] and Lawther [9]. Duckworth [4] showed that this list is complete for maximal rank subgroups.

In Theorem 1 we use the notational conventions just described, as well as the following. We write $X = L_i$ if X is conjugate to a Levi subgroup obtained by crossing off node i from the Dynkin diagram of G (we number the nodes of the Dynkin diagram as in [2]). The meaning of $X = L_{i_1, i_2}$ and $P = P_i$ is similar.

Theorem 1. *Let G be a simple algebraic group of type A_n, B_n, C_n or D_n, let X be a maximal rank reductive subgroup defined over $\mathbb{Z}$ and $P \neq G$ a parabolic subgroup. Then $X\backslash G/P$ is finite if and only if X is spherical or one of the following holds:*

(i) $G = A_n$,
  (a) $P \in \{P_1,\ P_n\}$,
  (b) $X = A_{n_1}A_{n_2}A_{n_3}T_2$ *and* $P = P_i$ *for some* i;

(ii) $G = B_n$, $X = A_{n_1}B_{n_2}T_1$ *and* $P \in \{P_1,\ P_n\}$;

(iii) $G = C_n$,
  (a) $X \in \{C_{n_1} \cdots C_{n_r},\ \ A_{n_1}C_{n_2} \cdots C_{n_r}T_1\}$ *and* $P = P_1$,
  (b) $X \in \{C_{n_1}C_{n_2}C_{n_3},\ \ C_{n_1}C_{n_2}T_1,\ \ A_{n_1}C_{n_2}T_1\}$ *and* $P = P_n$;

(iv) $G = D_n$,
  (a) $G = D_4$, $(X, P) \in \{(L_{2,3}, P_4),\ (L_{2,4},\ P_3)\}$,
  (b) $X \in \{A_{n_1}D_{n_2}T_1,\ \ A_{n_1}A_{n_2}T_2\}$ *and* $P = P_1$,

(c) $X \in \{D_{n_1}D_{n_2}D_{n_3},\ \ A_{n_1}D_{n_2}T_1\}$ *and* $P \in \{P_{n-1},\ \ P_n\}$.

2 History and preliminaries

The problem of identifying finite double coset collections has been studied by a variety of authors, usually in the context of some other problem. The spherical subgroups discussed above provide one example of this study. We briefly describe another example here, and refer the reader to [4] and [11] for fuller discussions.

Example 2.1. We describe irreducible finite orbit modules. Let V be a finite dimensional vector space and let X be a closed, connected subgroup of $G = \mathrm{GL}(V)$ such that X has finitely many orbits on V and V is irreducible under X. Such modules were classified by Kac [7] in characteristic 0 and by Guralnick, Liebeck, Macpherson and Seitz [6] in positive characteristic. This work was representation theoretic, however it is related to the double coset question studied in the present paper. Note that X is a reductive group and that $X\backslash\mathrm{GL}(V)/P_1$ is finite where P_1 is the stabilizer of a 1-space. The authors of [6] mentioned this connection with double cosets, they classified finiteness for $X\backslash\mathrm{GL}(V)/P_i$ where $1 \leq i \leq \dim V$ and they established Theorem 1 (i) using rather different arguments from those that appear in the present paper.

The following lemma provides basic reductions in our double coset question.

Lemma 2.2. *Let G be a group and let X and P be subgroups. Let Z be the center of G, suppose that $Z \leq P$ and let $\overline{X}$, $\overline{G}$ and $\overline{P}$ be the images of X, G and P, respectively, under the map $G \to G/Z$. Let K be a finite normal subgroup of G and let $\widehat{X}$, $\widehat{G}$ and $\widehat{P}$ be the images of X, G and P, respectively, under the map $G \to G/K$. Let $g, h \in G$. The following are equivalent:*

(i) $|X\backslash G/P| < \infty$,
(ii) $|\widehat{X}\backslash\widehat{G}/\widehat{P}| < \infty$,
(iii) $|\overline{X}\backslash\overline{G}/\overline{P}| < \infty$,
(iv) $|gXg^{-1}\backslash G/hPh^{-1}| < \infty$.

Let G be an algebraic group and X and P be closed subgroups. Denote by G°, X° and P° the identity components of G, X and P respectively. If $X\backslash G/P$ is finite then so is $X^\circ\backslash G^\circ/P^\circ$.

Proof. These statements can all be proven in an elementary fashion. The final statement uses only the fact that X° and P° are normal subgroups of finite index in X and P respectively. □

This Lemma justifies assumptions implicit in the statement and in the proof of Theorem 1. The final statement of the Lemma will be applied to recover finiteness results in $\mathrm{SO}(V)$ from arguments made involving $\mathrm{O}(V)$.

Remarks 2.3. We fix some notation for the rest of the paper. We let G be a classical algebraic group defined over a fixed algebraically closed field. If G is of type B_n, C_n or D_n we will, when convenient, assume that G is one of SO_{2n+1}, Sp_{2n} or SO_{2n} respectively. We will often replace $G = A_n$ with $G = \mathrm{GL}_n$. In all cases G has rank n (note that this means replacing P_n in A_n with P_{n-1} in GL_n). We denote by X a maximal rank reductive subgroup of G and by P a parabolic subgroup of G.

We fix some terminology since usage varies in the literature and refer the reader to [12] or [5] for further details. To each possibility for G we associate a natural module V and a bilinear and quadratic form (we take these forms to be identically zero if $G = \mathrm{GL}_n$). A subspace of V is totally singular if each form is identically zero on the subspace. If P is a parabolic subgroup then it equals the stabilizer of a (partial) flag of totally singular subspaces of V and we identify G/P as a collection of flags of totally singular subspaces. Thus, a spherical subgroup has a finite number of orbits on the set of flags of totally singular subspaces.

Let G equal SO_{2n}. Then two totally singular n-spaces are conjugate under G if and only if their intersection has odd codimension in each space. Thus, there exist two G-classes of totally singular n-spaces which we identify as G/P_{n-1} and G/P_n. If n is odd then L_{n-1} is conjugate to L_n under G, but this is not the case if n is even. These facts are relevant to Theorem 1 (iv).

Let V be the natural module for G. We write $V = V_1 \perp V_2$ if $V = V_1 \oplus V_2$ and each element of V_1 is orthogonal to each element of V_2. Given such a decomposition we define $\mathrm{Cl}(V_i)$ to equal $\mathrm{GL}(V_i)$, $\mathrm{O}(V_i)$ or $\mathrm{Sp}(V_i)$ as G equals, respectively, $\mathrm{GL}(V)$, $\mathrm{SO}(V)$ or $\mathrm{Sp}(V)$. We let $\mathrm{Cl}(V_i)^\circ$ be the connected identity component of $\mathrm{Cl}(V_i)$. If $\mathrm{Cl}(V_i) = \mathrm{O}(V_i)$ then $\mathrm{Cl}(V_i)^\circ = \mathrm{SO}(V_i)$ and otherwise $\mathrm{Cl}(V_i)^\circ = \mathrm{Cl}(V_i)$.

3 Finiteness

In this section we will prove those parts of Theorem 1 which assert finiteness. For the convenience of the reader we list in Table 2 the specific result which covers each case.

Table 2: Finiteness Cases

G	Cases	Proof
A_n	all cases	Corollary 3.6
B_n	the case with $P = P_1$	Corollary 3.6
	the case with $P = P_n$	Corollary 3.11
C_n	all cases with $P = P_1$	Corollary 3.6
	all cases with $P = P_n$	Corollary 3.9
D_n	$G = D_4$, all cases	Corollary 3.13
	$(X, P) = (A_{n_1} D_{n_2} T_1,\ P_1)$	Corollary 3.6
	$(X, P) = (A_{n_1} A_{n_2} T_2,\ P_1)$	Lemma 3.12
	$(X, P) \in \{D_{n_1} D_{n_2} D_{n_3},\ A_{n_1} D_{n_2} T_1\} \times \{P_{n-1},\ P_n\}$	Corollary 3.9

The next Lemma is not used immediately, but we place it here to preserve the line of argument later.

Lemma 3.1. *Let $G = \mathrm{SO}_{2n}$ and let $X = \mathrm{GL}_n$. Then X acts transitively upon the set of definite* 1*-spaces in V and has a finite number of orbits upon the set of all* 1*-spaces in V.*

Proof. Let N_1 be the stabilizer of a definite 1-space in G. Then the first claim is equivalent to having $G = XN_1$. In this form the first claim is proven in [10]. The second claim follows from the facts that GL_n is spherical in G and that every 1-space is either singular or definite. $\square$

Remark 3.2. We fix some notation for the next few Lemmas. Let G be one of $\mathrm{GL}(V)$, $\mathrm{O}(V)$, $\mathrm{SO}(V)$, or $\mathrm{Sp}(V)$ and fix a decomposition $V = V_1 \perp V_2$ such that $X = X_1 X_2$ for subgroups $X_i \leq \mathrm{Cl}(V_i)$. Let P be a maximal parabolic subgroup of G and identify G/P as a collection of totally singular subspaces in V. For $i \in \{1,\ 2\}$ let $\pi_i : V \to V_i$ be the natural projection.

Lemma 3.3. *Let the notation be as in Remark 3.2. In addition let β and φ be, respectively, the bilinear and quadratic forms associated with G.*

(i) *Let $u,\ v,\ x,\ y \in V$ such that $\beta(u, v) = \beta(x, y) = 0$. Then $\beta(\pi_1 u,\ \pi_1 v) = \beta(\pi_1 x,\ \pi_1 y)$ if and only if $\beta(\pi_2 u,\ \pi_2 v) = \beta(\pi_2 x,\ \pi_2 y)$.*
(ii) *Let u and x be two singular vectors with $\varphi(\pi_1 u) = \varphi(\pi_1 x)$. Then $\varphi(\pi_2 u) = \varphi(\pi_2 x)$.*

Proof. Decompose u as $u = \pi_1 u + \pi_2 u$, decompose v, x, and y similarly, and use the fact that V_1 and V_2 are orthogonal to each other. $\square$

Lemma 3.4. *Let the notation be as in Remark 3.2 with the additional assumptions that $G = \mathrm{Cl}(V)$ and $X_2 = \mathrm{Cl}(V_2)$. Two totally singular subspaces of the same dimension are conjugate under X if and only if their projections to V_1 and intersections with V_1 are simultaneously conjugate under X_1.*

The proof of this statement uses Witt's Theorem applied to X_2, so, if G is an orthogonal group, one cannot replace $\mathrm{Cl}(V_2) = \mathrm{O}(V_2)$ with $\mathrm{Cl}(V_2)^\circ = \mathrm{SO}(V_2)$. We will use Lemma 2.2 to translate finiteness results to SO_n. We note that finiteness results do not always translate between O_n and SO_n in an obvious fashion. For example, the collection $L_{2,3}\backslash \mathrm{SO}_8/P_4$ is finite whereas $L_{2,3}\backslash \mathrm{O}_8/P_4$ is infinite.

Proof. It is easy to see that if two subspaces are conjugate under X, then their projections to V_1 and intersections with V_1 are simultaneously conjugate under X_1.

Conversely, let W and W' be totally singular subspaces of the same dimension such that $x_1(W \cap V_1,\, \pi_1 W) = (W' \cap V_1,\, \pi_1 W')$ for some $x_1 \in X_1$. Replacing W with $x_1 W$ we may assume that $(W \cap V_1,\, \pi_1 W) = (W' \cap V_1,\, \pi_1 W')$. Note that $\dim W \cap V_2 = \dim W' \cap V_2$. Define the following dimensions:

$$\begin{aligned} a &= \dim W \cap V_1 = \dim W' \cap V_1, \\ c &= \dim W \cap V_2 = \dim W \cap V_2, \\ b &= \dim W - a - b = \dim W' - a - b. \end{aligned}$$

We will pick bases for W and W' as follows:

$$\begin{array}{rllll} W: & w_1, \ldots, w_a, & w_{a+1}, \ldots, w_{a+b}, & w_{a+b+1}, \ldots, w_{a+b+c}, \\ W': & w_1, \ldots, w_a, & w'_{a+1}, \ldots, w'_{a+b}, & w'_{a+b+1}, \ldots, w'_{a+b+c}. \end{array}$$

We start by picking a basis $w_1, \ldots, w_a$ of $W \cap V_1 = W' \cap V_1$. Extend this with elements $v_{a+1}, \ldots, v_{a+b}$ to a basis of $\pi_1 W = \pi_1 W'$. For each $i \in \{a+1, \ldots, a+b\}$ pick $w_i \in W$ and $w'_i \in W'$ such that $\pi_1 w_i = \pi_1 w'_i = v_i$. Let $w_{a+b+1}, \ldots, w_{a+b+c}$ and $w'_{a+b+1}, \ldots, w'_{a+b+c}$ be bases for $W \cap V_2$ and $W' \cap V_2$ respectively. Then each of $\{w_{a+1}, \ldots, w_{a+b+c}\}$ and $\{w'_{a+1}, \ldots, w'_{a+b+c}\}$ is a linearly independent set, whence each of $\{\pi_2 w_{a+1}, \ldots, \pi_2 w_{a+b+c}\}$ and $\{\pi_2 w'_{a+1}, \ldots, \pi_2 w'_{a+b+c}\}$ is a linearly independent set.

Let $\widetilde{x}_2$ be the linear map from the subspace $\langle \pi_2 w_{a+1}, \ldots, \pi_2 w_{a+b+c}\rangle$ to $\langle \pi_2 w'_{a+1}, \ldots, \pi_2 w'_{a+b+c}\rangle$ which takes each $\pi_2 w_i$ to $\pi_2 w'_i$.

If $G = \mathrm{GL}(V)$ then one may extend $\widetilde{x}_2$ to an element $x_2 \in \mathrm{GL}(V_2) = X_2$. Note that $x_2 w_i = w'_i$ for each $i \in \{a+1, \ldots, a+b+c\}$. This finishes the proof for the case $G = \mathrm{GL}(V)$.

If $G \in \{\mathrm{O}(V),\ \mathrm{Sp}(V)\}$ we show that $\widetilde{x}_2$ is an isometry from the subspace $\langle \pi_2 w_{a+1}, \ldots, \pi_2 w_{a+b+c}\rangle$ to $\langle \pi_2 w'_{a+1}, \ldots, \pi_2 w'_{a+b+c}\rangle$. Once this is done, Witt's

Theorem implies that we may again extend $\widetilde{x}_2$ to $x_2 \in \mathrm{Cl}(V_2) = X_2$ and we will be finished. If $p \neq 2$ or if G is symplectic, then Lemma 3.3 (i) shows that $\beta(\pi_2 w_i, \pi_2 w_j) = \beta(\pi_2 w_i', \pi_2 w_j')$ for all $a+1 \leq i, j \leq a+b+c$, whence $\widetilde{x}_2$ is an isometry. If $p = 2$ and G is orthogonal, then Lemma 3.3 (ii) shows that $\varphi(\pi_2 w_i) = \varphi(\pi_2 w_i')$ for $a+1 \leq i \leq a+b+c$, whence $\widetilde{x}_2$ is an isometry. □

Corollary 3.5. *Let the notation be as in Remark* 3.2 *with the additional assumption that* X_2 *equals* $\mathrm{Cl}(V_2)$ *or* $\mathrm{Cl}(V_2)^\circ$. *Then* $X\backslash G/P$ *is finite in the following cases:*

(i) X_1 *has a finite number of orbits upon the set of* 1*-spaces in* V_1 *and* $P = P_1$,
(ii) $G = \mathrm{GL}(V)$, X_1 *has a finite number of orbits upon the set of all flags in* V_1 *and* $P = P_i$ *for some* i,
(iii) $G = \mathrm{GL}(V) = \mathrm{GL}_n$, X_1 *has a finite number of orbits upon the set of subspaces of* V_1 *with codimension* 1 *and* $P = P_{n-1}$.

Proof. By Lemma 2.2 it suffices to prove Corollary 3.5 with the assumption that $G = \mathrm{Cl}(V)$ and $X = \mathrm{Cl}(V_2)$. By Lemma 3.4 it suffices to show that X_1 has a finite number of orbits upon the set $\{(W \cap V_1,\, \pi_1 W) \mid W \in G/P\}$. Note that $W \cap V_1 \leq \pi_1 W$ is a flag. It is easy to verify in each case that X_1 has a finite number of orbits on the set. □

Corollary 3.6. *The double coset collection* $X\backslash G/P$ *is finite in the following cases:*

(i) $G = \mathrm{GL}_n$, (a) X *has no additional restrictions (other than our standing assumptions) and* $P \in \{P_1, P_{n-1}\}$ *or* (b) $X = \mathrm{GL}_{n_1}\mathrm{GL}_{n_2}\mathrm{GL}_{n_3}$ *and* $P = P_i$ *for some* i,
(ii) $G = \mathrm{SO}_{2n+1}$, $X = \mathrm{GL}_{n_1}\mathrm{SO}_{2n_2+1}$ *and* $P = P_1$,
(iii) $G = \mathrm{Sp}_{2n}$, $X \in \{\mathrm{GL}_{n_1}\mathrm{Sp}_{2n_2}\cdots\mathrm{Sp}_{2n_r},\ \mathrm{Sp}_{2n_1}\cdots\mathrm{Sp}_{2n_r}\}$ *and* $P = P_1$,
(iv) $G = \mathrm{SO}_{2n}$, $X = \mathrm{GL}_{n_1}\mathrm{SO}_{2n_2}$ *and* $P = P_1$.

Proof. In each case let V be the natural module for G and fix a decomposition $V = V_1 \perp V_2$ such that $X = X_1X_2$ with $X_1 \leq \mathrm{Cl}(V_1)$ and $X_2 = \mathrm{Cl}(V_2)^\circ$.

For case (i)(b) note that $X_1 = \mathrm{GL}_{n_1}\mathrm{GL}_{n_2}$ is a spherical subgroup of $\mathrm{GL}_{n_1+n_2} = \mathrm{Cl}(V_1)$ and apply Corollary 3.5 (ii). For case (i)(a) apply Corollary 3.5 (i) if $P = P_1$, apply Corollary 3.5 (iii) if $P = P_{n-1}$, and, in both cases, induct on r.

Let $G = \mathrm{Sp}_{2n}$. By Corollary 3.5 (i) it suffices to show that X_1 has finitely many orbits on totally singular 1-spaces in V (note that all 1-spaces are totally singular in this case). This is immediate if X_1 equals GL_{n_1} or $\mathrm{Sp}_{2n_1}\mathrm{Sp}_{2n_2}$ since these subgroups are spherical in $\mathrm{Cl}(V_1)^\circ$. The general case follows by induction on r.

If G is orthogonal, then $X_1 = \mathrm{GL}_{n_1}$ and the result follows by combining Lemma 3.1 and Corollary 3.5 (i). $\square$

Lemma 3.7. *Let the notation be as in Remark 3.2 with the additional assumption that V is a symplectic or orthogonal space. Let W be a maximal totally singular subspace of V and let $(\pi_i W)^\perp$ be the perpendicular space taken within V_i. If $\dim V$ is even then $\dim(\pi_i W)^\perp/(W \cap V_i)$ equals 0. If $\dim V$ is odd then $\dim(\pi_i W)^\perp/(W \cap V_i)$ equals 0 or 1.*

Proof. Since V_1 is orthogonal to V_2 it is easy to show that $W \cap V_i \le (\pi_i W)^\perp$. We have $\dim W$ equals n and $\dim V$ equals $2n$ or $2n{+}1$. Set $a_i = \dim W \cap V_i$. For i equal to 1 and 2 the inequality $\dim(\pi_i W)^\perp \ge \dim W \cap V_i$ becomes, respectively, $\dim V_1 - (n - a_2) \ge a_1$ and $\dim V_2 - (n - a_1) \ge a_2$. If $\dim V$ is even, then the sum of these last two inequalities is an equality; if $\dim V$ is odd, then the sum of the left sides is 1 greater than the sum of the right sides. $\square$

Corollary 3.8. *Let the notation be as in Remark 3.2 with the additional assumptions that G is not $\mathrm{GL}(V)$, that X_2 equals $\mathrm{Cl}(V_2)^\circ$ or $\mathrm{Cl}(V_2)$ and that $P = P_n$. Then $X\backslash G/P$ is finite in the following cases:*

(i) *$\dim V$ is even and X_1 has a finite number of orbits on the set of totally singular subspaces of V_1,*
(ii) *$\dim V$ is odd and X_1 has a finite number of orbits on the set of pairs of subspaces (W_1, W_2) such that $W_1 \le W_2 \le V_1$, W_1 is totally singular, $W_1 \le W_2^\perp$ and $\dim(W_2/W_1) \le 1$.*

Proof. By Lemma 2.2 we may assume that $G = \mathrm{Cl}(V)$ and $X_2 = \mathrm{Cl}(V)$. By Lemma 3.4 it suffices to show that X_1 has a finite number of orbits upon the set $\{(W \cap V_1, \pi_1 W) \mid W \in G/P\}$. Given subspaces $W, W' \le V$ and $x_1 \in X_1$ we have $x_1 \pi_1 W = \pi_1 W'$ if and only if $x_1(\pi_1 W)^\perp = (\pi_1 W')^\perp$, where we take the perpendicular space within V_1. Thus, it suffices to show that X_1 has a finite number of orbits on the set $\{(W \cap V_1,\ (\pi_1 W)^\perp) \mid W \in G/P\}$. By Lemma 3.7 we see that $\{(W \cap V_1,\ (\pi_1 W)^\perp) \mid W \in G/P\}$ is a subset (or may be identified with a subset) of one of the sets given in the statement of Corollary 3.8. $\square$

Corollary 3.9. *The double coset collection $X\backslash G/P$ is finite in the following cases:*

(i) *$G = \mathrm{SO}_{2n}$, $X \in \{\mathrm{SO}_{2n_1} \mathrm{SO}_{2n_2} \mathrm{SO}_{2n_3},\ \mathrm{GL}_{n_1} \mathrm{SO}_{2n_2}\}$ and $P \in \{P_{n-1},\ P_n\}$.*
(ii) *$G = \mathrm{Sp}_{2n}$, $X \in \{\mathrm{Sp}_{2n_1} \mathrm{Sp}_{2n_2} \mathrm{Sp}_{2n_3},\ \mathrm{GL}_{n_1} \mathrm{Sp}_{2n_2},\ \mathrm{Sp}_{2n_1} \mathrm{Sp}_{2n_2} T_1\}$ and $P = P_n$.*

Proof. Let V be the natural module of G and fix a decomposition $V = V_1 \perp V_2$ so that $X = X_1 X_2$ with $X_1 \leq \mathrm{Cl}^\circ(V_1)$ and $X_2 = \mathrm{Cl}(V_2)^\circ$. Then X_1 is a spherical subgroup of $\mathrm{Cl}(V_1)^\circ$ whence the conclusion follows from Corollary 3.8 (i). $\square$

Lemma 3.10. *Let $G = \mathrm{SO}_{2n}$, let $X = \mathrm{GL}_n \leq \mathrm{SO}_{2n}$ and let V be the natural module for G. Let $\langle v \rangle$ be a definite 1-space, let $X_{\langle v \rangle}$ be the stabilizer in X of $\langle v \rangle$ and let $\widetilde{X}_{\langle v \rangle}$ be the connected component of the group induced by $X_{\langle v \rangle}$ in $\mathrm{SO}(\langle v \rangle^\perp)$. Then $\widetilde{X}_{\langle v \rangle}$ is a spherical subgroup of $\mathrm{SO}(\langle v \rangle^\perp)$.*

Proof. By Lemma 3.1 we may calculate $X_{\langle v \rangle}$ where v is any definite vector. Let $V = E \oplus F$ such that E and F are totally singular and X is the stabilizer in G of this decomposition. Let $\{e_1, \ldots, e_n\} \subset E$ and $\{f_1, \ldots, f_n\} \subset F$ be dual bases. Let $v = e_1 + f_1$ and let GL_{n-1} denote the subgroup of X which stabilizes the subspaces $\langle e_2, \ldots, e_n \rangle$ and $\langle f_2, \ldots, f_n \rangle$ and acts trivially upon e_1 and f_1. Then (an isomorphic image of) GL_{n-1} is a subgroup of $\widetilde{X}_{\langle v \rangle}$ which proves the claim since $\mathrm{SO}(\langle v \rangle^\perp) = \mathrm{SO}_{2(n-1)+1}$.

$\square$

Corollary 3.11. *Let $G = \mathrm{SO}_{2n+1}$, $X = \mathrm{GL}_{n_1} \mathrm{SO}_{2n_2+1}$, and $P = P_n$. Then $X \backslash G / P$ is finite.*

Proof. Let V be the natural module of G and fix a decomposition $V = V_1 \perp V_2$ so that $X = X_1 X_2$ with $X_1 = \mathrm{GL}_{n_1} \leq \mathrm{Cl}^\circ(V_1)$ and $X_2 = \mathrm{SO}_{2n_2+1} = \mathrm{Cl}(V_2)^\circ$. By Corollary 3.8 (ii) it suffices to show that X_1 has a finite number of orbits on the set of pairs of subspaces (W_1, W_2) such that $W_1 \leq W_2 \leq V_1$, W_1 is totally singular, $W_1 \leq W_2^\perp$ and $\dim(W_2/W_1) \leq 1$. We may partition this set into two subsets according as W_2 is, or is not, totally singular. Since X_1 is spherical in $\mathrm{SO}(V_1)$ we see that X_1 has finitely many orbits upon the subset where W_2 is totally singular.

Every pair (W_1, W_2) where W_2 is not totally singular can be rewritten as $(W_1, W_1 \perp \langle v \rangle)$ where v a definite vector in V_1. Thus it suffices to show that X_1 has a finite number of orbits upon pairs $(W_1, \langle v \rangle)$ such that v is a definite vector in V_1 and $W_1 \leq \langle v \rangle^\perp$ is totally singular (where this perpendicular space is taken in V_1). By Lemma 3.1, X_1 acts transitively upon definite 1-spaces, whence it suffices to fix v and show that the stabilizer in X_1 of $\langle v \rangle$ has a finite number of orbits on totally singular subspaces in $\langle v \rangle^\perp$. This follows from Lemma 3.10. $\square$

Lemma 3.12. *Let $G = \mathrm{SO}_{2n}$, $X = \mathrm{GL}_{n_1} \mathrm{GL}_{n_2}$ and $P = P_1$. Then $X \backslash G / P$ is finite.*

Proof. Let V be the natural module of G and fix a decomposition $V = V_1 \perp V_2$ so that $X = X_1 X_2$ with $X_i = \mathrm{GL}_{n_i} \leq \mathrm{Cl}^\circ(V_i)$ for each i.

Let $\pi_i : V \to V_i$ be the natural projection. We wish to show that X has a finite number of orbits on the set $\{\langle v\rangle \mid v \in V \text{ is singular}\}$. By Lemma 3.1 each X_i has finitely many orbits on 1-spaces in V_i. Thus it suffices to show, for each singular 1-space $\langle v\rangle$ that X has finitely many orbits on the set of singular 1-spaces whose projections to V_1 and V_2 are conjugate to $\pi_1\langle v\rangle$ and $\pi_2\langle v\rangle$ respectively. Thus, it suffices to fix an arbitrary singular 1-space $\langle v\rangle$, let $v_i = \pi_i v$ and show that X has a finite number of orbits on the set

$$\{\langle v_1 + \alpha v_2\rangle \mid \alpha \in k,\ \alpha \neq 0,\ v_1 + \alpha v_2 \text{ is singular}\}, \tag{1}$$

where k is the ground field. Since $v_1 + \alpha v_2$ is singular, we have $\varphi(v_1) + \alpha^2\varphi(v_2) = 0$, where φ is the quadratic form. Thus, v_1 is definite if and only if v_2 is. If v_1 and v_2 are both definite then $\alpha^2 = -\varphi(v_1)/\varphi(v_2)$, whence there are at most two singular 1-spaces of the form $\langle v_1 + \alpha v_2\rangle$.

It suffices now to assume that v_1 and v_2 are singular and show that X_1 has finitely many orbits on the set in Equation 1. Let $V_2 = E_2 \oplus F_2$ where E_2 and F_2 are totally singular and X_2 stabilizes E_2 and F_2. Then $v_2 = e + f$ for some $e_2 \in E_2$, $f_2 \in F_2$. Since v_2 is singular this implies that $\beta(e, f) = 0$, where β is the bilinear form. Easy linear algebra shows that there exists $x_2 \in X_2$ with $x_2 e = \alpha e$ and $x_2 f = \alpha f$, whence $x_2\langle v_1 + v_2\rangle = \langle v_1 + \alpha v_2\rangle$. □

Recall that $L_{i,2}$ denotes a Levi subgroup as described just before Theorem 1.

Corollary 3.13. *Let $G = D_4$ and $(X, P) \in \{(L_{2,3},\ P_4),\ (L_{2,4},\ P_3)\}$. Then $X\backslash G/P$ is finite.*

Proof. This follows most easily from applying the graph automorphism of order three to other cases which have been proven finite. For instance $(X, P) = (L_{2,3}, P_4)$ follows from Corollary 3.9 applied to $(X, P) = (L_{1,2}, P_3)$ (with $L_{1,2} = D_1 D_2 D_2$) or from Lemma 3.12 applied to $(X, P) = (L_{2,4}, P_1)$. □

The reader who wishes for an instructive, though somewhat painful, exercise can prove this Corollary using geometric arguments about subspaces of the natural module of D_4.

4 Infiniteness

We begin by stating a result which gives infiniteness in many cases.

Theorem 2 ([4, Theorem 1.3]). *If $X\backslash G/P$ is finite then X or L is a spherical subgroup of G.*

Table 3: Infiniteness cases

G	(X, P)	Proof
A_n	$\{A_{n_1}A_{n_2}A_{n_3}A_{n_4}T_3\} \times \{P_i \mid 2 \le i \le n-1\}$	Lemma 4.4
B_n	$\{B_{n_1}D_{n_2}D_{n_3},\ A_{n_1}A_{n_2}T_2\} \times \{P_1,\ P_n\}$	Lemma 4.5
C_n	$\{A_{n_1}A_{n_2}C_{n_3}T_2\} \times \{P_1,\ P_n\}$	Lemma 4.6
	$(C_{n_1}C_{n_2}C_{n_3}C_{n_4}\ (n_i \ge 1),\ P_n)$	Lemma 4.6
	$(A_{n_1}C_{n_2}C_{n_3}T_1\ (n_i \ge 1),\ P_n)$	Lemma 4.6
D_n	$(D_{n_1}D_{n_2}D_{n_3},\ P_1)$	Lemma 4.7
	$\{D_{n_1}D_{n_2}D_{n_3}D_{n_4}\} \times \{P_{n-1},\ P_n\}$	Lemma 4.7
	$\{A_{n_1}D_{n_2}D_{n_3}\ (n_1 \ge 1)\} \times \{P_{n-1},\ P_n\}$	Lemma 4.7
	$\{A_{n_1}A_{n_2}T_2\ (n_i \ge 1)\} \times \{P_{n-1},\ P_n\}$ with (G, X, P) not as in Theorem 1 (iv)(a)	Lemma 4.8

Remark 4.1. To finish the proof of infiniteness in Theorem 1 it suffices, by Theorem 2, to consider only those P such that L is spherical. Suppose that we have fixed such a P. Then it suffices to prove infiniteness for those X which are maximal subject to the condition that $X\backslash G/P$ is claimed to be infinite in Theorem 1.

In Table 3 we list those cases which need to be proven infinite, and indicate which result addresses each case. Recall that we allow the notation A_0, B_0, C_0 and D_1 unless otherwise noted (but we do not allow D_0).

Remark 4.2. For the remainder of this section we assume that X and L contain a common maximal torus, T. For a closed subgroup H which contains T we write $\Phi(H)$ for the root system of H defined using T. For a closed root subsystem φ of $\Phi(G)$ we set $\dim \varphi = \dim H/Z$ where H is a closed subgroup of G such that $\Phi(H) = \varphi$ and Z is the center of H (Theorem 3 also holds if we use $\dim \varphi = \dim H$ instead).

Theorem 3 ([4, Theorem 1.1, Lemma 3.3]). *For $i \in \{1, 2\}$ let L_i be a Levi subgroup containing T and Φ_i its root system. Assume that L_1 and L_2 are conjugate. If $\frac{1}{2}\dim \Phi_1 - \dim \Phi_1 \cap \Phi(X) - \frac{1}{2}\dim \Phi_2 \cap \Phi(L) > 0$ then $X\backslash G/P$ is infinite. In particular infiniteness holds in the following cases:*

(i) *Φ_1 and Φ_2 are of type B_2, $\Phi_1 \cap \Phi(X) = \emptyset$ and $\Phi_2 \cap \Phi(L)$ is of type A_1.*

(ii) *Φ_1 and Φ_2 are of type A_3 or D_3, $\Phi_1 \cap \Phi(X) = \emptyset$ and Φ_2 is of type A_1A_1 or D_2.*

Remarks 4.3. We offer comments which help simplify the proofs of Lemmas 4.4, 4.5, 4.6 and 4.7.

(a) All Levi subgroups of type B_2 are conjugate and, unless $G = D_n$, all Levi subgroups of type A_3 are conjugate. Thus, to apply Theorem 3 one often only has to verify that Φ_1, $\Phi_1 \cap \Phi(X)$, Φ_2 and $\Phi_2 \cap \Phi(L)$ are of the required type. We will construct each Φ_i by giving a base α, β, ... and setting Φ_i equal to all the $\mathbb{Z}$-linear combinations of α, β, ... which are in $\Phi(G)$. (b) Let $\Delta(G)$ and $\widetilde{\Delta}(G)$ be the Dynkin diagram and extended Dynkin diagram of G respectively. Label the nodes of $\Delta(G)$ using $\alpha_1, \ldots, \alpha_n$ as in [2]. In each of the following Lemmas we will assume that $\Delta(X)$ has been produced from $\Delta(G)$ by the Borel-de Siebenthal algorithm [1]. Let $X = X_{n_1}X_{n_2}\cdots$ with each X_{n_i} equal to D_1, T_1 or a simple factor of X, as listed in Table 3, with n_i the rank of the factor. We take $\Delta(X_{n_1})$ equals to $\emptyset$ if X_{n_1} equals T_1 or D_1 and otherwise $\Delta(X_{n_1}$ equals the first n_1 nodes of $\Delta(G_1)$ or $\widetilde{\Delta}(G_1)$ as appropriate. We then repeat this procedure, starting with $\Delta(X_{n_2})$ and the last $n_2 + n_3 + \ldots$ nodes of $\Delta(G)$. This procedure determines $\Delta(G) - \Delta(X)$ which, in turn, provides an easy description of $\Phi(X)$. For example, suppose that $G = D_n$ and $X = D_{n_1}D_{n_2}$. Then $\Delta(G) - \Delta(X) = \{\alpha_{n_1}\}$ and $\Phi(X)$ equals all the roots in $\Phi(G)$ which have α_{n_1}-coefficient equal to 0 or ± 2. (c) Given α, $\beta \in \Delta(G)$ the path connecting α to β is the shortest such path and includes α and β. The sum over this path means the sum of each element of $\Delta(G)$ which is contained in the path. It is easy to check that such a sum is itself a root.

Lemma 4.4. *Let $G = A_n$, $X = A_{n_1}A_{n_2}A_{n_3}A_{n_4}T_3$ and $P \in \{P_i \mid 2 \le i \le n-1\}$. Then there exist Φ_1 and Φ_2 of type A_3 as in Theorem* 3 (ii).

Proof. We have $\Delta(G) - \Delta(X) = \{\alpha_{n_1+1},\ \alpha_{n_1+n_2+2},\ \alpha_{n_1+n_2+n_3+3}\}$. Let Φ_1 have root base given by α equal to α_{n_1+1}, β equal to the sum over the path connecting α_{n_1+2} to $\alpha_{n_1+n_2+2}$, and γ equal to the sum over the path connecting $\alpha_{n_1+n_2+3}$ to $\alpha_{n_1+n_2+n_3+3}$.

For L_i let Φ_2 have root base given by $\alpha = \alpha_{i-1}$, $\beta = \alpha_i$, and $\gamma = \alpha_{i+1}$. $\square$

Lemma 4.5. *Let $G = B_n$ and $(X, P) \in \{B_{n_1}D_{n_2}D_{n_3},\ A_{n_1}A_{n_2}T_2\} \times \{P_1, P_n\}$. Then there exist Φ_1 and Φ_2 of type B_2 as in Theorem* 3 (i).

Proof. One may proceed as in the proofs of Lemmas 4.4 and 4.8, or as in [4, Corollary 7.2 (ii)]. $\square$

Lemma 4.6. *Let $G = C_n$. If $(X, P) \in \{A_{n_1}A_{n_2}C_{n_3}T_2\} \times \{P_1, P_n\}$, then there exists $\Phi_1 = \Phi_2$ of type B_2 as in Theorem* 3 (i). *If $(X, P) \in \{C_{n_1}C_{n_2}C_{n_3}C_{n_4}$ $(n_i \ge 1)$, $A_{n_1}C_{n_2}C_{n_3}T_1$ $(n_i \ge 1)\} \times \{P_n\}$, then there exist Φ_1 and Φ_2 of type A_3 as in Theorem* 3 (ii).

Proof. One may proceed as in the proofs of Lemmas 4.4 and 4.8. □

Lemma 4.7. *Let G equal D_n. If (X, P) equals $(D_{n_1}D_{n_2}D_{n_3},\ P_1)$, or is in $\{A_{n_1}D_{n_2}D_{n_3}T_1\ (n_1 \geq 1),\ D_{n_1}D_{n_2}D_{n_3}D_{n_4}\} \times \{P_{n-1},\ P_n\}$, then there exists $\Phi_1 = \Phi_2$ of type A_3 or D_3 as in Theorem* 3 (ii).

Proof. One may proceed as in the proofs of Lemmas 4.4 and 4.8. □

Lemma 4.8. *Let $G = D_n$, $X = A_{n_1}A_{n_2}T_2$ $(n_i \geq 1)$, $P \in \{P_{n-1},\ P_n\}$. If $n = 4$ we assume that (X, P) is not equal to either $(L_{2,3},\ P_4)$ or $(L_{2,4},\ P_3)$. Then there exists Φ_1 and Φ_2 of type A_3 or D_3 as in Theorem* 3 (ii).

Proof. By our convention with subsystems of type A_{n_i} in D_n, we have that $X \in \{L_{i,n-1},\ L_{i,n}\}$ where $i = n_1 + 1$ satisfies $2 \leq i \leq n-2$.

Let $(X, P) = (L_{i,n},\ P_n)$. Let $\Phi_1 = \Phi_2$ have root base given by α equal to the sum over the path connecting α_1 to α_{n-1}, β equal to α_n, and γ equal to the sum over the path connecting α_2 to α_{n-2}.

By symmetry the conclusion holds also when $(X, P) = (L_{i,n-1},\ P_{n-1})$. This leaves the cases $(X, P) \in \{(L_{i,n-1},\ P_n),\ (L_{i,n},\ P_{n-1})\}$. If n is odd then $L_{i,n-1}$ is conjugate to $L_{i,n}$, whence the conclusion holds. It remains to prove the existence of Φ_1 and Φ_2 when $n \geq 6$ is even. If necessary we replace X by a conjugate to assume that $i \leq n-3$. Let $(X, P) = (L_{i,n-1}, P_n)$ with $2 \leq i \leq n-3$. Let $\Phi_1 = \Phi_2$ have root base given by α equal to the sum over the path connecting α_i to α_{n-3}, $\beta = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$, and γ equal to the sum over the path connecting α_{i-1} to α_{n-2}. By symmetry the conclusion also holds when $(X, P) = (L_{i,n}, P_{n-1})$. □

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